U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
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OFFICE NOTE 6

A TWO-LEVEL MODEL WITH EFFECTS OF VERTICAL VARIETY ADUCTION, SINGLES TO RAIN, AND VARIABLE STATIC STABILITY

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p=400 or 450 mb

I $\omega_0 = \frac{10^3 g m^2}{4d^2}\left[\frac{l}{f}\mathbb{J}(\hat{\psi}, P_g) - \frac{1}{f}\frac{P_g - 600}{P}\mathbb{J}(h, P_g)\right]$ $\quad 2^{24}\omega_0 = 2^{20}\hat{\omega}_0 \quad \hat{\omega}_0 = 2^{-4}\omega_0$

II $\nabla^2\hat{\chi} = -\frac{\hat{\omega}_0}{10^3 p}\frac{d^2}{m^2}2^3 \cdot 2^{-30}$ $\quad \chi = 2^{30}\hat{\chi}$

III $\mathbb{V}_x = \frac{g l m}{2fd}\frac{\partial\hat{\psi}}{\partial y} + \frac{2^{30} m}{2d}\frac{\partial\hat{\chi}}{\partial x}$ $\quad \mathbb{V}_y = \frac{g l m^2}{2fd}\frac{\partial\psi}{\partial x} + \frac{2^{30} m}{2d}\frac{\partial\hat{\chi}}{\partial y}$ $\quad \psi = \frac{g l}{f}\hat{\psi}$

$\chi = 2^{30}\hat{\chi}$

IV $\frac{\partial\sigma^2}{\partial t} = -C\left[\frac{g l m^2}{4fd^2}\mathbb{J}(\hat{\psi}, \sigma^2) + \frac{2^{30}m^2}{d^2}A(\hat{\chi}, \sigma^2) - C_2\frac{6\hat{\omega}_0\sigma^2 \cdot 2^4}{10^3 \cdot 2P}\right.$ $\quad \omega_0 = 2^4\hat{\omega}_0$

$P_{mb} = 10^3 P_{cgs}$

V $\eta = \frac{g l m^2}{4fd^2}\nabla^2\hat{\psi} + f$

VI $\frac{m^2}{d^2}\nabla^2\left(\frac{\partial h}{\partial t}\right) - \frac{2f\eta}{\sigma^2}\frac{\partial h}{\partial t} = \frac{2f\eta}{\sigma^2}\left[\frac{g l m^2}{4fd^2}\mathbb{J}(\hat{\psi}, h) + \frac{2^{30}m^2}{4d^2}A(\hat{\chi}, h)\right)$

$- \left[\frac{g l m^2}{4fd^2}\mathbb{J}\left(\hat{\psi}, \frac{m^2}{d^2}\nabla^2 h\right) + \frac{2^{30}m^2}{4d^2}A\left(\hat{\chi}, \frac{m^2}{d^2}\nabla^2 h\right)\right] - \frac{m^2}{4d^2}\mathbb{J}(h, \eta) - \frac{2^4 f\eta\hat{\omega}_0}{10^3 g P}$

VII $\omega = \frac{10^3 g P}{\sigma^2}\left[\frac{\partial h}{\partial t} + \frac{g l m^2}{4^3 d}\mathbb{J}(\hat{\psi}, h) + \frac{2^{30}m^2}{4d^2}A(\hat{\chi}, h)\right]$

VIII $\frac{g l m^2}{fd^2}\nabla^2\frac{\partial\psi}{\partial t} = -B\left[\frac{g l m^2}{4fd^2}\mathbb{J}(\hat{\psi}, \eta) + \frac{2^{30}m^2}{4d^2}A(\hat{\chi}, \eta)\right] + \frac{g w m^2}{fPd^2 10^3}\nabla^2 h + \frac{2^4\eta\hat{\omega}_0}{10^3 2P}$

1. Derive Z_{600} from Z_{400} and Z_{850} by means of

expression $Z_{600} = 0.53792\, Z_{850} + 0.46208\, Z_{400} + C$

Provide for C having a field. When Z is in feet, C = 270 ft.
[80% of standard atmosphere static stability has been used]
Print out for inspections

2. Compute the two components u_0 and v_0 of

$$V_0 = \mathbb{K} \times \nabla\psi - \frac{P_g - 600}{P}\,\frac{g}{f}\,\mathbb{K} \times \nabla h$$

$\mathbb{V}_0$?

where ψ is the 600 mb stream function, P_g the pressure at the ground,
$h = Z_{400} - Z_{850}$ and $P = 450$ mb
Print out for inspection the two components u_0 & v_0
in appropriate units

3. $P = 450$ in equ I
4. $P = 400$ mb in the remaining equ
5. $\omega_0 = -g\rho W_0 = 1.3\, W_0$ where W_0 is in cm/sec
$1\text{ mb} = 10^3$ cgs

6. In equ 6 replace $\mathbb{V}\cdot\nabla(\nabla^2 h)$ by $\frac{2f}{g}\mathbb{V}\cdot\nabla\left(\frac{g}{2f}\nabla^2 h\right)$

7. In equ 4 $C\frac{\omega_0\sigma^2}{2P}$ may be replaced by $C\frac{\omega\sigma^2}{2P}$

8. The contribution of the various terms (forcings) in the prognostic equations will for each time step be dumped on tape for later use (separate prognosis)

9. Make provisions for running the prognostic equations without 1) the ω_0 term 2) the $\nabla^2 h$ terms

10 program equ IV in the form

$$\frac{\partial}{\partial t}(\sigma^2) + C_1\,\mathbb{V}\cdot\nabla(\sigma^2) + C_2\,\frac{\omega_0\sigma^2}{2P} = 0$$

where C_1 & C_2 are two constants, yet not determined

STD 850 = 4,780 ft

STD 400 = 23,564 ft

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A Two-Level Model with Effects of Vertical Vorticity Convection, Irregular Terrain, and Variable Static Stability.

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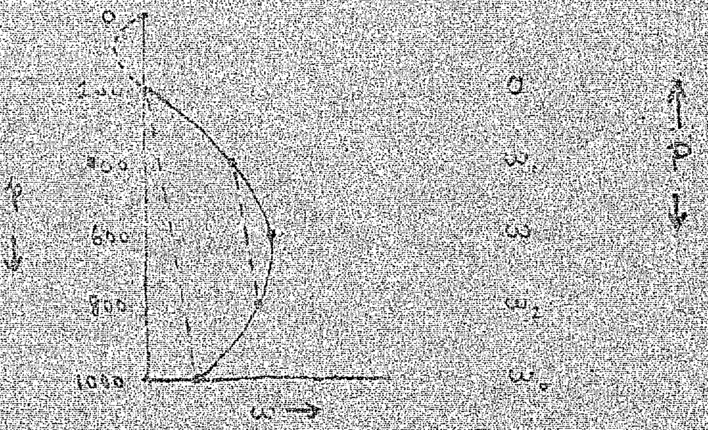
In this model, the ω-profile is assumed to have a smooth shape, as shown below. We suppose that ω vanishes



at 200 mb (see T.M. # 11), and that ω vanishes at 1000 mb over flat terrain. In general, ω_0 is not zero, so that the ω-profile is not symmetrical around 600 mb. As sketched in the figure above, we assume that asymmetrical profiles are such that

$$\frac{\omega_2 - \omega_1}{P} = \frac{\omega_0}{2P}$$

Applying the vorticity equation at the 400 and 800 mb surfaces, and replacing $\partial\eta/\partial p$ and $\partial\omega/\partial p$ by finite-differences,

$$\frac{\partial \eta_1}{\partial t} + V_1 \cdot \nabla \eta_1 + \omega_1 \frac{(\eta_2 - \eta_1)}{P} - \frac{\eta_1 \omega}{P} = 0$$

$$\frac{\partial \eta_2}{\partial t} + V_2 \cdot \nabla \eta_2 + \omega_2 \frac{(\eta_2 - \eta_1)}{P} - \eta_2 \frac{(\omega_0 - \omega)}{P} = 0$$

where the subscripts 1 and 2 refer to conditions at the 400 and 800 mb surfaces, respectively. Subtracting the second of these equations from the first, and rearranging

$$\frac{\partial}{\partial t}\left(\frac{\eta_1-\eta_2}{2}\right) + \mathbf{V}^*\cdot\nabla\eta' + \mathbf{V}'\cdot\nabla\eta^* + \left(\frac{\eta_2-\eta_1}{2}\right)\left(\frac{\omega_1-\omega_2}{p}\right)$$

$$- \frac{\omega}{2p}(\eta_1+\eta_2) + \frac{\eta_2\omega_0}{2p} = 0$$

in which

$$\eta' = \frac{\eta_1-\eta_2}{2} \qquad \eta^* = \frac{\eta_1+\eta_2}{2}$$

Thus, since $\omega_2-\omega_1 \doteq \omega_0/2$,

$$\frac{\partial\eta'}{\partial t} + \mathbf{V}^*\cdot\nabla\eta' + \mathbf{V}'\cdot\nabla\eta^* - \frac{\eta^*\omega}{p} + \frac{\eta^*\omega_0}{2p} = 0 \qquad (1)$$

We must introduce the geostrophic approximation in computing η' and $\mathbf{V}'$ in the equation above. By definition

$$\eta' = \frac{\eta_1-\eta_2}{2} = \frac{g}{2f}\left(\nabla^2 z_1 - \nabla^2 z_2\right) = \frac{g}{2f}\nabla^2 h$$

$$\mathbf{V}' = \frac{\mathbf{V}_1-\mathbf{V}_2}{2} = \frac{g}{2f}\,\mathbf{k}\times\left(\nabla z_1 - \nabla z_2\right) = \frac{g}{2f}\,\mathbf{k}\times\nabla h$$

where h is the thickness between the 400 and 800 mb surfaces. Thus, substituting in Eq. (1),

$$\frac{\partial}{\partial t}\nabla^2 h + v^*\cdot\nabla(\nabla^2 h) + J(h,\eta^*) - \frac{2f\eta^*\omega}{gP} + \frac{f\eta^*\omega_0}{gP} = 0 \quad (2)$$

Now, the adiabatic equation may be written as

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + v\cdot\nabla\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho\theta}\frac{\partial\theta}{\partial p}\right)\omega = 0$$

where all quantities apply at 600 mb. But, from the hydrostatic equation

$$\frac{1}{\rho} = \frac{g(z_1 - z_2)}{p} = \frac{gh}{p}$$

Thus, substituting in the equation above,

$$\frac{\partial h}{\partial t} + v\cdot\nabla h + \frac{P}{\rho\theta}\frac{\partial\theta}{\partial p}\cdot\frac{\omega}{g} = 0$$

$$\text{or} \quad \frac{\omega}{gP} = \frac{1}{\sigma^2}\left[\frac{\partial h}{\partial t} + v\cdot\nabla h\right] \quad (3)$$

$$\text{where} \quad \sigma^2 = -\frac{P^2}{\rho\theta}\frac{\partial\theta}{\partial p} = \frac{P^2}{\rho^2}\left(\frac{1}{g\theta}\frac{\partial\theta}{\partial z}\right)$$

Finally, introducing this result into Eq. (2), and identifying $\mathbb{V}^*$ with $\mathbb{V}$ (the 600 mb wind), and η^* with η,

$$\frac{\partial}{\partial t}\nabla^2 h - \frac{2f\eta}{\sigma^2}\frac{\partial h}{\partial t} = \frac{2f\eta}{\sigma^2}\mathbb{V}\cdot\nabla h - \mathbb{V}\cdot\nabla(\nabla^2 h)$$

$$- J(h,\eta) - \frac{f\eta\,\omega_0}{gP} \qquad (4)$$

This is one of the two basic prognostic equations.

Applying the vorticity equation at 600 mb,

$$\frac{\partial\eta}{\partial t} + \mathbb{V}\cdot\nabla\eta + \omega\frac{(\eta_2-\eta_1)}{P} - \frac{\eta\omega_0}{2P} = 0$$

Thus, computing $\eta_2-\eta_1$ from the geostrophic wind,

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathbb{V}\cdot\nabla(\nabla^2\psi + f) - \frac{g\omega}{fP}\nabla^2 h - \frac{\eta\omega_0}{2P} = 0 \qquad (5)$$

This is the other basic prognostic equation.

It is now suppose that h and ψ are given initially. We enumerate the quantities that are needed to compute h and ψ at the next time stage. They are:

η

∇^2

$\mathbb{V}$

ω_0

ω

η is, of course, $(\nabla^2\psi + f)$. The quantity ω_0 is ω at 1000 mb. Approximately

$$\omega_0 = \mathbb{V}_0 \cdot \nabla p_g$$

where $\mathbb{V}_0$ is the wind at the ground surface, and p_g is the pressure at the ground surface. By linear extrapolation

$$\mathbb{V}_0 = \mathbb{k} \times \nabla\psi + (p_g - 600)\frac{\partial \mathbb{V}}{\partial p}$$

$$\frac{\partial \mathbb{V}}{\partial p} = \frac{\mathbb{V}_2 - \mathbb{V}_1}{P} = -\frac{2\mathbb{V}'}{P} = -\frac{g}{fP}\,\mathbb{k} \times \nabla h$$ whence

$$\mathbb{V}_0 = \mathbb{k} \times \nabla\psi - \left(\frac{p_g - 600}{P}\right)\frac{g}{f}\,\mathbb{k} \times \nabla h$$

7.

$$\omega_0 = J(\psi, p_z) - \left(\frac{p_z - 600}{p}\right)\frac{f}{g} J(h, p_z) \qquad (6)$$

In order that there be no systematic effect due to the inclusion of only the stretching effect of convergence in downslope flow, we put

$$\mathbb{V} = \mathbb{k}\times\nabla\psi + \nabla\chi$$

$$\nabla\cdot\mathbb{V} = \nabla^2\chi = -\frac{\omega_0}{2P} \qquad (7)$$

The last quantity required to complete the system is σ^2, which is proportional to $\frac{\partial\theta}{\partial p}$. We have, <u>in θ coordinates</u>,

$$\frac{d}{dt}\left(\frac{\partial p}{\partial\theta}\right) = \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\theta}\right) + \mathbb{V}\cdot\nabla_\theta\left(\frac{\partial p}{\partial\theta}\right)$$

$$= \frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial t}\right) + \frac{\partial}{\partial\theta}(\mathbb{V}\cdot\nabla_\theta p) - \frac{\partial\mathbb{V}}{\partial\theta}\cdot\nabla_\theta p$$

But in quasi-geostrophic flow $\frac{\partial\mathbb{V}}{\partial\theta}$ is very nearly perpendicular to $\nabla_\theta p$, so that

$$\frac{d}{dt}\left(\frac{\partial p}{\partial\theta}\right) \simeq \frac{\partial}{\partial\theta}\left(\frac{dp}{dt}\right)$$

$$\frac{\partial\theta}{\partial p}\,\frac{d}{dt}\left(\frac{\partial p}{\partial\theta}\right) = \frac{\partial\omega}{\partial p}$$

$$\frac{d}{dt}\ln\frac{\partial p}{\partial\theta} = \frac{\partial\omega}{\partial p}$$

$$-\frac{d}{dt}\ln\frac{\partial\theta}{\partial p} = \frac{\partial\omega}{\partial p}$$

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial p}\right) + \frac{\partial\theta}{\partial p}\,\frac{\partial\omega}{\partial p} = 0$$

Thus, in _p- coordinates_,

$$\frac{\partial}{\partial t}\left(\frac{\partial\theta}{\partial p}\right) + V\cdot\nabla\frac{\partial\theta}{\partial p} + \frac{\partial}{\partial p}\left(\omega\frac{\partial\theta}{\partial p}\right) = 0$$

We now average each term of this equation with respect to p, integrating from 200 to 1000 mb. Approximately then

$$\frac{\partial s}{\partial t} + V\cdot\nabla s + \frac{\omega_0 s_0}{2P} = 0$$

where s is a value of $\partial\theta/\partial p$, representative of the entire troposphere, and s_0 is its value at 1000 mb. Thus, if we assume that $s_0 = .6\,s$?

$$\frac{\partial s}{\partial t} + V\cdot\nabla s + \frac{\text{[illegible]}\; s}{2P} = 0$$

Finally, since the remaining factors of σ^2 vary much more slowly (percentagewise) than $\frac{\partial \theta}{\partial p}$,

$$\frac{\partial}{\partial t}(\sigma^2) + V \cdot \nabla \sigma^2 + \frac{6\omega_0 \sigma^2}{2P} = 0 \qquad (8)$$

Once σ^2 and $\frac{\partial h}{\partial t}$ have been computed, it is a straightforward matter to compute ω.

Summary of equations in order of solution

$$\omega_0 = J(\psi, p_g) - \left(\frac{p_g - 600}{P}\right)\frac{g}{f}J(h, p_g) \qquad \mathrm{I}$$

$$\nabla^2\chi = -\frac{\omega_0}{2P} \qquad \mathrm{II}$$

$$\mathbb{V} = \mathbb{k}\times\nabla\psi + \nabla\chi \qquad \mathrm{III}$$

$$\frac{\partial}{\partial t}(\sigma^2) + \mathbb{V}\cdot\nabla(\sigma^2) + \frac{\omega_0\sigma^2}{2P} = 0 \qquad \mathrm{IV}$$

$$\eta = \nabla^2\psi + f \qquad \mathrm{V}$$

$$\nabla^2\left(\frac{\partial h}{\partial t}\right) - \frac{2f\eta}{\sigma^2}\frac{\partial h}{\partial t} = \frac{2f\eta}{\sigma^2}\mathbb{V}\cdot\nabla h - \mathbb{V}\cdot\nabla(\nabla^2 h)$$

$$- J(h, \eta) - \frac{f\eta\omega_0}{gP} \qquad \mathrm{VI}$$

$$\omega = \frac{gP}{\sigma^2}\left(\frac{\partial h}{\partial t} + \mathbb{V}\cdot\nabla h\right) \qquad \mathrm{VII}$$

$$\nabla^2\left(\frac{\partial\psi}{\partial t}\right) + \mathbb{V}\cdot\nabla\eta - \frac{g\omega}{fP}\nabla^2 h - \frac{\eta\omega_0}{2P} = 0 \qquad \mathrm{VIII}$$

ψ = streamfunction at 600 mb

$h = z_{400} - z_{800}$

[illegible]

Begin with ψ, h, σ^2

1. Compute ω_0 from I
2. Compute χ by inverting II, using output of 1 and 2
3. Compute V from III, using output of 2

5. Solve VI for $\frac{\partial h}{\partial t}$, using output of 1 and 3
6. Compute ω from VII, using output of 3 and 5
7. Solve VIII for $\frac{\partial \psi}{\partial t}$, using output of 1, 3, and 6
8. Compute σ^2 from IV, using output of 1 and 3